ANNALS OF PHYSICS 115, 153–171 (1978)

Solitons as Infinite-Constituent Bound States

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Received February 1, 1978

We construct soliton states as quantum corrections to the coherent states constructed from classical solutions of nonlinear field equations. The corresponding field operator applied to the vacuum state allows us to discuss physical features of the (one soliton plus various mesons) states.

1. INTRODUCTION

A great deal of work has been done recently on solitons, both from the classical and quantal viewpoint [1]. The latter is especially important if the basic program of constructing some of the subnuclear particles as soliton states of other particles is to be effected. In order to do that, two features are necessary. First it is necessary to have a system of nonlinear equations which have some possibility of properly representing the fundamental particles from which the solitons are to be constructed. On the other hand there must also have been developed a powerful enough technology so that the properties of the resulting solitons can be read off. The first question is far from being answered, though progress is undoubtedly being made through gauge field theories and the discovery of the new particles. It is to the second question that this paper is devoted.

There have been various techniques developed to quantise solitons [2]. These appear to have given rise to somewhat of a mystique about solitons, as if they were objects of a very different sort from the particles of which they were composed. The purpose of this paper is to attempt to dispel this mystique and give a very down-to-earth view of solitons. As the title of this paper indicates, the soliton is to be shown as being a bound state of the elementary particles of which it is constituted. Naturally the particular form of bound state is very different from that state usually considered as acceptable for such an epithet to be applied. We have to deal here with coherent states, so that there may be an infinite number of constituents in the soliton state. We will find that the existence of a topologically conserved quantum number corresponds to the average number of constituent particles in the soliton state being infinite. Moreover the probability of decay of such a state into any finite number of its constituents is zero in this case.

The soliton we are interested in investigating is not a pure coherent state. We start our analysis by determining, in Section 2, a general form the soliton state could

0003-4916/76/1151–0153$05.00/0

take, and then construct an approximation method for obtaining this state. In the following section we consider static solitons in one space and one time dimension at the lowest level. We then turn to the one meson plus one soliton sector, and discuss the various quantities of interest, such as the soliton form factor. This is extended in Section 5 to the two meson plus one soliton sector. The Schrödinger field representation is used in Section 6 to obtain an approximation to the various one soliton plus meson states to arbitrary orders of perturbation in the higher derivatives of the potential, evaluated at the classical solution. We then extend the analysis to include time-dependent solutions to the classical equations, for moving solitons in the lowest approximation. Various features of the results of our analysis are discussed in the final section.

2. Construction of the Soliton State

We consider a theory in d space-dimensions and take a field ϕ which may have spin and internal spin labels, which will not be considered explicitly. We suppose that the field satisfies canonical commutation relation at equal time and is described by the Hamiltonian

$$H = H_0 + H_1, \tag{1}$$

where H_0 is a free field Hamiltonian quadratic in ϕ and H_1 is a polynomial or transcendental function in ϕ. We shall assume here for simplicity that

$$\begin{aligned} H_0 &= \int \frac{1}{2} (\dot{\phi}^2 + (\nabla\phi)^2)\, d^d\mathbf{x} \\ H_1 &= \int \frac{1}{g^2}\, U(g\phi)\, d^d\mathbf{x} \end{aligned} \tag{2}$$

for some coupling constant g. Then the field equations will be

$$\ddot{\phi} - \nabla^2\phi = U'(g\phi) \tag{3}$$

and the CCR's are

$$[\dot{\phi}(\mathbf{x}, t), \phi(\mathbf{y}, t)]_- = -\hbar\, \delta^3(\mathbf{x} - y). \tag{4}$$

We are interested in eigenstates of H determined by possible classical solutions of (3). In particular we wish to consider such solutions $\phi_c(\mathbf{x}, t)$ that are $O(g^{-1})$ as $g \to 0$, so are nonperturbative. To obtain the corresponding eigenstate we will use the coherent state operator

$$U(f, t) = \exp\left[\frac{i}{\hbar} M(f, t)\right] \tag{5}$$

$$M(f, t) = \int d^d\mathbf{x}[f \overleftrightarrow{\partial}_0 \phi], \tag{6}$$

where the integration in $M(f, t)$ is at the value of time equal to t and $\hat{f} = f - \phi_0$, where $\phi_0 = \langle 0 \mid \phi \mid 0 \rangle$. If we use the CCR's (4), then

$$\begin{aligned} U(f, t)\, \phi(x, t)\, U^{-1}(f, t) &= \hat{\phi}(x, t) + f(x, t) \\ U(f, t)\, \dot{\phi}(x, t)\, U^{-1}(f, t) &= \dot{\phi}(x, t) + \dot{f}(x, t), \end{aligned} \tag{7}$$

where

$$\langle 0 \mid \hat{\phi} \mid 0 \rangle = 0, \; \phi = \hat{\phi} + \phi_0 .$$

Thus

$$\begin{aligned} UHU^{-1} &= H(\hat{\phi} + f) \\ &= E(f) + O(\hat{\phi}), \end{aligned} \tag{8}$$

where $O(\hat{\phi})$ denotes terms linear or of higher powers in $\hat{\phi}$ and $\dot{\phi}$, and $E(f)$ is the classical energy for f. Then

$$HU^{-1}(\phi_c, t) \mid 0 \rangle = U^{-1} UHU^{-1} \mid 0 \rangle = E(\phi_c)\, U^{-1} \mid 0 \rangle + \text{quantum corrections}. \tag{9}$$

Therefore $U^{-1}(\phi_c, t) \mid 0 \rangle$ is an approximation to the soliton state for which we are searching. It is time-dependent, but since the energy $E(\phi_c)$ is not we can choose the state defined at any value of t.

Let us assume that at any time t the fields $\{\phi(\mathbf{x}, t), \dot{\phi}(\mathbf{x}, t)\}_{\mathbf{x} \in R^d}$ form a complete set. This has well-known difficulties when ultraviolet divergences are present, but we will neglect those for the present discussion. We attempt to obtain a better approximation to the above soliton state by choosing the expression (taken at $t = 0$ for convenience):

$$\begin{aligned} \mid f, \{f_{ij}\} \rangle = U^{-1}(f, 0) \Big\{ & f_0 + \int [f_{10}(x)\, \dot{\phi}(x) + f_{01} \hat{\phi}(x)]\, dx \\ & + \int [f_{20}(x_1, x_2)\, \dot{\phi}(x_1)\, \dot{\phi}(x_2) + f_{21}(x_1, x_2)\, \dot{\phi}(x_1)\, \hat{\phi}(x_2) \\ & + f_{22}(x_1, x_2)\, \hat{\phi}(x_1)\, \hat{\phi}(x_2)]\, dx_1\, dx_2 + \cdots \\ & + \int [f_{n0}(x_1 \cdots x_n)\, \dot{\phi}(x_1) \cdots \dot{\phi}(x_n) + \cdots \\ & + f_{nn}(x_1 \cdots x_n)\, \hat{\phi}'x_1) \cdots \hat{\phi}(x_n)]\, dx_1 \cdots dx_n + \cdots \Big\} \mid 0 \rangle \\ = U^{-1}(f, 0)\, & V(\{f_{ij}\}) \mid 0 \rangle. \end{aligned} \tag{10}$$

We assume that the constant f_0 and the functions f_{ij} can be found such that the series (10) converges and that it represents an eigenstate of the Hamiltonian H of (1) and (2) with eigenvalue $E(f_{ij}\})$:

$$H \mid f, \{f_{ij}\} \rangle = E(f, \{f_{ij}\}) \mid f, \{f_{ij}\} \rangle. \tag{11}$$

By Eq. (8), (11) becomes

$$H(\hat{\phi} + f)\, V \mid 0 \rangle = E\, V \mid 0 \rangle. \tag{12}$$

We may rewrite (12) somewhat by using that

$$H(\phi) \mid 0\rangle = 0$$

$$H(\hat{\phi} + f) = H(\phi) + \int [\dot{\phi}\dot{f} + \hat{\phi}'f' + U(\hat{\phi} + f) - U(\phi) - U(f)]\, dx + E(f)$$

$$[H(\hat{\phi} + f), \hat{\phi}(x)]_- = i\hbar(\dot{\phi}(x) + \dot{f}(x)) \tag{13}$$

$$[H(\hat{\phi} + f), \dot{\phi}(x)]_- = i\hbar(-\hat{\phi}'' - f'' + U'(\hat{\phi} + f))(x) \tag{14}$$

so that (12) becomes

$$\left\{[H(\hat{\phi} + f), V]_- + V \int [\dot{\phi}\dot{f} - \hat{\phi}f'' + U(\hat{\phi} + f) - U(\phi) - U(f)]\, dx\right\} \mid 0\rangle$$
$$= [E - E(f)]V \mid 0\rangle. \tag{15}$$

We attempt to solve (12) or (15). If we write

$$H(\hat{\phi} + f) = (\tfrac{1}{2}\dot{\phi}^2 + \dot{\phi}\dot{f}) + (\tfrac{1}{2}\phi'^2 + \phi'f' + U(\hat{\phi} + f)), \tag{16}$$

we need only concern ourselves with reordering the second term on the r.h.s. of (16) in its action on $V \mid 0\rangle$. This reordering must be chosen to reproduce the order of $\dot{\phi}$ and ϕ chosen in the definition (10) of V, that is with $\dot{\phi}$'s always to the left of ϕ's. We perform such reordering by the formulas

$$e^{\lambda\hat{\phi}(x)} \prod_{j=1}^{t} \dot{\phi}(x_j)\, e^{-\lambda\hat{\phi}(x)} = \prod_{j=1}^{t} [\dot{\phi}(x_j) + \lambda i\hbar\delta(x - x_j)]$$

$$\left[\hat{\phi}^m(x), \prod_{j=1}^{t} \dot{\phi}(x_j)\right]_- = d^m/d\lambda^m \left[\prod_{j=1}^{t} [\dot{\phi}(x_j) + \lambda i\hbar\delta(x - x_j)]\, e^{\lambda\hat{\phi}(x)}\right]\Bigg|_{\lambda=0}$$

$$= \sum_{s=0}^{m} \binom{m}{s} \sum \prod_{l=1}^{t-s} \dot{\phi}(x_{j_l}) \prod_{k=1}^{s} i\hbar\delta(x - x_{i_k})\, \hat{\phi}^{m-s}(x) \tag{17}$$

where the summation in the last expression in (17) is over all choices of $j_1 \cdots j_{t-s}$ and $i_1 \cdots i_s$ which are distinct from each other and lie between l and t.

We may now solve (12), using (17), to give the set of equations

$$\begin{aligned}
&[E - E(f)] f_{n,r}(x_1 \cdots x_{n-r}, y_1 \cdots y_r) \\
&\quad = S\dot{f}(x_1) f_{n-1,r}(x_2 \cdots x_{n-r}, y_1 \cdots y_r) \\
&\qquad + \tfrac{1}{2} S\delta(x_1 - x_2) f_{n-2,r}(x_3 \cdots x_{n-r}, y_1 \cdots y_r) \\
&\qquad - S f_{n-1,r-1}(x_1 \cdots x_{n-r}, y_1 \cdots y_{r-1}) f''(y_r) \\
&\qquad - \tfrac{1}{2} S f_{n-2,r-2}(x_1 \cdots x_{n-r}, y_1 \cdots y_{r-2})\, \delta''(y_{r-1} - y_r) \\
&\qquad - i\hbar(n - r + 1) \int f''(x) f_{n+1,r}(x_1 \cdots x_{n-r}, y_1 \cdots y_r)\, dx
\end{aligned}$$

$$- 2i\hbar(n - r + 1)S \frac{\partial^2}{\partial y_1^2} f_{n,r-1}(x_1 \cdots x_{n-r} y_1, y_2 \cdots y_{n-r})$$

$$+ S \sum_{m \geqslant 1} \int dx \frac{u_m}{m!} \sum_{s=0}^{m} \binom{m}{s} (i\hbar)^s f_{n+2s-m,r+s-m}(x_1 \cdots x_{n-r}, x \cdots x, y_1 \cdots y_{r+s-m})$$

$$\times \prod_{i=r+s-m}^{r} \delta(y_i - x) \cdot \binom{n + 2s - m}{s} \tag{18}$$

where S denotes symmetrisation over the appropriate set of x and y variables (separately):

$$Sf(x_1 \cdots x_n, y_1 \cdots y_m) = \frac{1}{n!\, m!} \sum_{\pi_1, \pi_2} f(x_{\pi_1(1)} \cdots x_{\pi_1(m)}, y_{\pi_2(1)} \cdots y_{\pi_2(m)})$$

with summation over all permutations π_1 and π_2 of $(1 \cdots n)$ and $(1 \cdots m)$ respectively, $u_m = U^{(n)}(f)$, and x occurs s times in the integrand of the last term in (18). We note that the system of equation (18) is infinitely coupled, but can be solved by iteration in powers of $\hbar$ since the terms on the r.h.s. of (18) not involving $\hbar$ all involve $f_{n',r'}$ with $n' + r' < n + r$.

We may also equate coefficients of various products of $\hat{\phi}$'s and $\hat{\dot{\phi}}$'s in Eq. (15) though the resulting equations are more complicated than (18) and we will not write them explicitly here. However, we note that the first term on the left of (15) is of order $\hbar$ compared to the other terms, so an iteration scheme in $\hbar$ will still be possible. We will return to that shortly.

3. The Static Two-Dimensional Soliton in Lowest Approximation

In order to develop the theory further, especially in understanding the physical nature of the resulting state $U^{-1}V \mid 0\rangle$ introduced in the last section let us consider the lowest order of approximation for the soliton constructed on a static solution $\phi_c(x)$ in one space and one time dimension to the classical field equation

$$-\phi_c'' + U'(\phi_c) = 0. \tag{18a}$$

We remarked in the previous section that $\mid \phi_c, t\rangle = U^{-1}(\phi_c, t) \mid 0\rangle$ is an approximation to the soliton state, with energy $E(\phi_c) + O(\hbar)$. We note that this state also has the form-factor interpretation[1]:

$$\begin{aligned}\langle \phi_c, t \mid \phi(x, t) \mid \phi_c, t\rangle &= \langle 0 \mid U(\phi_c, t)\phi(x, t)\, U^{-1}(\phi_c, t) \mid 0\rangle \\ &= \langle 0 \mid \hat{\phi}(x, t) + \phi_c(x) \mid 0\rangle = \phi_c(x).\end{aligned}$$

Thus the state $\mid \phi_c, t\rangle$ would be interpreted as a soliton at the origin with meson form-factor $\phi_c(x)$, as is now traditional [1].

We may translate the vacuum state $| 0\rangle$ to represent a soliton at any point y in space by means of the operator

$$M(f, t, y) = \int dx[\dot{f}(x, t)\, \overleftrightarrow{\partial}_0 \phi(x + y, t)]$$

so that with

$$U(f, t, y) = \exp[(i/\hbar)M(f, t, y)]$$

we have

$$U(f, t, y)\phi(x, t)\, U^{-1}(f, t, y) = \hat{\phi}(x, t) + f(x - y, t).$$

Thus the state

$$| \phi_c , t, y\rangle = U^{-1}(\phi_c , t, y) \,| 0\rangle$$

will have energy $[E(\phi_c) + O(\hbar)]$ and form factor

$$\langle\phi_c , t, y \mid \phi(x, t) \mid \phi_c , t, y\rangle = \phi_c(x - y).$$

More generally we can construct a soliton state at y from the general one at the origin in (10) by application of the translation operator exp $[(i/\hbar)(Py)]$, in the form

$$(UV)_y = e^{-iPy/\hbar} UV\, e^{-iPy/\hbar}.$$

We note that such soliton states are absolutely stable against meson decay, since we may calculate the overlap between such a state and a state with a finite number of free mesons in it by choosing $t = \pm\infty$ in the soliton state and using the asymptotic condition to replace the Heisenberg field ϕ by a free field. We then use that

$$\langle 0 \mid U(f_1 , \infty)\, U^{-1}(f_2 , \infty) \mid 0\rangle = \exp\left[-\frac{1}{2}\int dk \mid \tilde{f}_1 - \tilde{f}_2 \mid^2 \omega_k\right]$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the Fourier transforms of f_1 and f_2 and $\omega_k = (k^2 + \mu^2)^{1/2}$. If $f_1 = 0$ and $f_2 = \hat{\phi}_c$, the transition probability we are investigating will depend on $\int dk\, \omega_k \mid \tilde{\hat{\phi}}_c \mid^2$. This will be infinite if $\hat{\phi}_c$ is nonzero at infinity, since then

$$\infty = \int \mid \hat{\phi}_c \mid^2 dx = \int \mid \tilde{\hat{\phi}}_c \mid^2 dk < \int \omega_k \mid \tilde{\hat{\phi}}_c \mid^2 dk.$$

The resulting transition probabilities will all be zero, so that the existence of a topological conservation law $(\hat{\phi}_c(\pm\infty) \neq 0)$ corresponds to the soliton being a stable bound state. It will be composed of an infinite number of elementary constituents (hence the title of the article) with momentum distribution function $\tilde{\hat{\phi}}_c(k)$. We note that this remark on numbers of constituents also applies to the more general state (10), either at the origin or elsewhere.

4. Static Two-Dimensional Soliton in the One-Meson Approximation

Let us now turn to solving (15) in the lowest nontrivial approximation, thus keeping only f_0, f_{10}, and f_{01} in the expression V of (10). For this approximation

$$V = f_0 + \int [f_{10}(x)\, \dot{\phi}(x) + f_{01}(x)\, \hat{\phi}(x)]\, dx$$

the first term on the right of (15) is

$$i\hbar \int \{f_{10}[-\phi'' - \phi_c'' + U'(\hat{\phi} + \phi_c)] - f_{01}\dot{\phi}\}\, dx.$$

Thus equating coefficients of 1, $\dot{\phi}$ and $\hat{\phi}$ on both sides of (15), and neglecting the second term of (15), since $U(\phi_0) = U'(\phi_0) = 0$, so it is $0(\hat{\phi}^2)$ we obtain

$$[E - E(\phi_c)] f_0 = +i\hbar \int f_{10}(x)[-\phi_c''(x) + U'(\phi_c)]\, dx \tag{19}$$

$$[E - E(\phi_c)] f_{10}(x) = -i\hbar f_{01}(x) \tag{20}$$

$$[E - E(\phi_c)] f_{01}(x) = i\hbar[-f_{10}''(x) + U''(\phi_c) f_{10}(x)]. \tag{21}$$

Since ϕ_c satisfies the classical equation (18a) the r.h.s. of (19) vanishes, so that either $E = E(\phi_c)$ or $f_0 = 0$. If we choose the former of these possibilities then $f_{01} = 0$ and

$$-f_{10}'' + U''(\phi_c) f_{10} = 0, \tag{22}$$

so that $f_{10} \propto \phi_c'$. On the other hand if $f_0 = 0$ and we denote $E - E(\phi_c)$ by $\lambda\hbar$ ($\lambda \neq 0$) then (20) and (21) combine to give

$$-f_{10}'' + U''(\phi_c) f_{10} = \lambda^2 f_{10} \tag{23}$$

$$f_{01} = i\lambda f_{10}\, . \tag{24}$$

Equation (23) is the stability equation for the classical solution ϕ_c of the original nonlinear equation (18a). Its solutions for $\lambda \neq 0$ are, for a wide class of potentials, the set ϕ_{λ_i} with discrete positive eigenvalues λ_i ($1 \leqslant i \leqslant N$) and the continuum ϕ_{λ_k} with the continuum $\lambda_k > \mu = [U''(\phi_0)]^{1/2}$, where $U(\phi_0) = U'(\phi_0) = 0$. We have thus determined the eigenstates $|\,\phi_c\,, \phi_c'\rangle$, $|\,\phi_c \phi_{\lambda_i}\rangle$ and $|\,\phi_c \phi_{\lambda_k}\rangle$ of H with eigenvalues $E(\phi_c)$, $E(\phi_c) + \hbar\lambda_i$ and $E(\phi_c) + \hbar\lambda_k$, respectively. We interpret these as a correction to the original static soliton state, a set of N excited solitons and the original soliton state with an added meson of momentum k (where $\lambda_k = (k^2 + \mu^2)^{1/2}$).

We note certain features of these results. First there has been no infrared difficulty associated with the translational mode ϕ_c'. Indeed it has come to play its role alongside ϕ_c very naturally at the next order of approximation in V. Second we may calculate the meson form factor in the modified soliton state, or in one of the soliton excited states ϕ_{λ_i}. We have the general result

$$\langle \phi_c, f_0, f_{10}, f_{01} \mid \phi(x) \mid \phi_c, f_0, f_{10}, f_{01} \rangle$$

$$= \langle 0 \mid \left[f_0 + \int (f_{10}^* \dot{\phi} + f_{01}^* \hat{\phi}) \right] [\hat{\phi}(x) + \phi_c(x)] \times \left[f_0 + \int [f_{10} \dot{\phi} + f_{01} \hat{\phi}] \right] \mid 0 \rangle$$

$$= f_0{}^2 \phi_c(x) + f_0 \int \left[f_{10}^*(y) \frac{\partial}{\partial y_0} + f_{01}^*(y) \right] W(y y_0 x x_0) \mid_{x_0 = y_0 = 0} dy + \text{h.c.}$$

$$+ \int \left[f_{10}^*(y) \frac{\partial}{\partial y_0} + f_{01}^{*\prime} y) \right] \left[f_{10}(z) \frac{\partial}{\partial z_0} + f_{01}(z) \right] W(y y_0 x x_0 z z_0) \mid_{x_0 = y_0 = z_0} dx\, dy\, dz$$

where $W(x\, x_0, \cdots) = \langle 0 \mid \hat{\phi}(x\, x_0) \cdots \mid 0 \rangle$ are the associated Wightman functions. Thus we may express these form factors either as corrections to $\phi_c(x)$ (when $\lambda = 0$, $f_{01} = \phi_c'$, $f_{10} = 0$) or directly by means of the nonsoliton Wightman functions and the excited soliton classical wavefunctions.

We can also evaluate the matrix elements of $\phi(x)$ between the 1 soliton state and the 1-meson + 1-soliton state as

$$\langle \phi_c \mid \phi(x) \mid \phi_c, \phi_{\lambda_k} \rangle = \langle 0 \mid [\hat{\phi}(x) + \phi_c(x)] \int [f_{10}(y)\, \dot{\phi}(y) + f_{01}{}' y)\, \hat{\phi}(y)] \mid 0 \rangle\, dy$$

$$\propto \int dy\, \phi_{\lambda_k}(y) \left[\frac{\partial}{\partial y_0} + i\lambda_k \right] W(x, 0, y, y_0) \mid_{y_0 = 0} dy \qquad (25)$$

where the constant of proportionality is determined from the particular continuum function. We may evaluate (25) by the single particle approximation to $W(x\, x_0\,;\, y\, y_0)$, giving

$$\langle \phi_c \mid \phi(x) \mid \phi_c, \phi_{\lambda_k} \rangle \propto \phi_{\lambda_k}(x) \qquad (26)$$

again fitting with the standard interpretation [1] of the continuum wavefunctions.

We can note more generally at this point that the matrix element of any product of meson fields in the 1-soliton sector can always be expressed, by (10), as a sum of Wightmans functions of the meson fields in the nonsoliton sector multiplied by the appropriate wavefunctions f_{ij}. Thus analysis of the soliton sector can be completely reduced to that of the nonsoliton sector and that of solving for the bound state expressions (10). For we can see that

$$\langle \phi_c, \{f_{ij}\} \mid \prod_i \phi(x_i) \mid \phi_c, \{f_{ij}\} \rangle = \langle 0 \mid V(\{f_{ij}\}) \prod_i [\hat{\phi}(x_i) + \phi_c(x_i)]\, V(\{f_{ij}\}) \mid 0 \rangle$$

from which the above result follows on explicit substitution of $V(\{f_{ij}\})$ from (10).

We can also use the soliton states as part of a complete sum of states to obtain, for example, the expression for the meson propagator in the soliton sector as

$$\langle \phi_c \mid \hat{\phi}(x_1)\, \hat{\phi}(x_2) \mid \phi_c \rangle = \phi_c(\mathbf{x}_1)\, \phi_c(\mathbf{x}_2) + \sum_{\lambda \neq 0} \langle \phi_c \mid \hat{\phi}(x_1) \mid \phi_c \phi_\lambda \rangle \langle \phi_c \phi_\lambda \mid \hat{\phi}(x_2) \mid \phi_c \rangle.$$

Using the result (26) we find

$$\langle \phi_c \mid T(\hat{\phi}(x_1 x_{10})\, \hat{\phi}(x_2 x_{20})) \mid \phi_c \rangle = \phi_c(x_1)\, \phi_c(x_2) + \sum_{\lambda \neq 0} [\alpha(\phi_\lambda)]^{+2}\, e^{-i\lambda |x_{10} - x_{20}|} \phi_\lambda(x_1)\, \phi_\lambda(x_2) \tag{27}$$

where $\alpha(\phi_\lambda)$ is the appropriate constant of proportionality in (26). We note that we may also introduce an effective meson field $\phi_s(x)$ in the 1-soliton sector defined from the creation and annihilation operators a_λ^+, a_λ which create or annihilate the excited soliton state or the 1-meson plus 1-soliton state ϕ_λ from the 1-soliton state ϕ_c:

$$| \phi_c , \phi_\lambda \rangle = a_\lambda^+ \mid \phi_c \rangle.$$

Then

$$\phi_s(x) = \phi_c(x) + \sum_\lambda \phi_\lambda(x)\, a_\lambda$$

so that

$$\langle \phi_c \mid \phi_s(x) \mid \phi_c \phi_\lambda \rangle = \phi_\lambda(x).$$

This agrees with our result (26), though we have already seen that such effective fields are not necessary in order for meson–soliton processes to be calculated. We have also justified the recent analysis of Steinmann [4].

5. The Two-Meson Approximation

We continue our analysis of Eq. (15) by including the two-meson states in V. Thus we take

$$V = f_0 + \int (f_{10}\dot{\phi} + f_{01}\hat{\phi})\, dx$$
$$+ \int [f_{20}(xy)\, \dot{\phi}(x)\, \dot{\phi}(y) + f_{21}(xy)\, \dot{\phi}(x)\, \hat{\phi}(y) + f_{22}(xy)\, \hat{\phi}(x)\, \hat{\phi}(y)]\, dx\, dy$$

so that

$$[H(\hat{\phi} + f), V] = i\hbar \int \{f_{10}(x)[-\phi''(x) - f''(x) + U'(\hat{\phi} + f)] - f_{01}(x)\, \dot{\phi}(x)\}\, dx$$
$$+ \int dx \int dy \{f_{20}(x, y)[i\hbar' - \phi''(x) - f''(x) + U'(\hat{\phi} + f))\, \dot{\phi}(y)$$
$$+ i\hbar\dot{\phi}(x)(-\phi''(y) - f''(y) + U'(\hat{\phi} + f))]$$
$$+ f_{21}(x, y)[i\hbar(-\phi''(x) - f''(x) + U'(\hat{\phi} + f))\, \hat{\phi}(y) - i\hbar\dot{\phi}(x)\, \dot{\phi}(y)]$$
$$- i\hbar f_{22}(x, y)[\dot{\phi}(x)\, \hat{\phi}' y) + \hat{\phi}(x)\, \dot{\phi}(y)]\}.$$

We also use that

$$U(\hat{\phi} + f) - U(\hat{\phi} + \phi_0) - U(f) - \hat{\phi} f'' = \tfrac{1}{2}\hat{\phi}^2[U''(f) - U''(\phi_0)]$$

if $f(x)$ satisfies the classical equation (3). Then equating coefficient in (15) of the various terms 1, $\dot{\phi}(x)$, $\hat{\phi}(x)$, $\dot{\phi}(x)\dot{\phi}(y)$, $\dot{\phi}(x)\hat{\phi}(y)$, $\hat{\phi}(x)\hat{\phi}(y)$, we obtain the equations

$$
\begin{aligned}
[E - E(f)]f_0 &= -\hbar^2 \int dx\, K_x(f) f_{20}(x, x) + \hbar^2 \int dx\, f_{22}(x, x) \\
[E - E(f)]f_{10}(x) &= -i\hbar f_{01}{}'x) \\
[E - E(f)]f_{01}(x) &= i\hbar K_x(f) f_{10} - \hbar^2 f_{20}(x, x)\, U'''(f) \\
[E - E(f)]f_{20}(x, y) &= -i\hbar f_{21}(x, y) \\
[E - E(f)]f_{21}(x, y) &= i\hbar[K_x(f) + K_y(f)]f_{20}(x, y) - 2i\hbar f_{22}(x, y) \\
[E - E(f)]f_{22}(x, y) &= \tfrac{1}{2}i\hbar[K_x(f) + K_y(f)]f_{21}(x, y) \\
&\quad + \tfrac{1}{2}\delta(x - y)\{f_0[U''(f) - U''(\phi_0)] \\
&\quad + i\hbar f_{10}(x)\, U'''(f) - \hbar^2 U^{(4)}(f) f_{20}(x, x)\}
\end{aligned}
\tag{28}
$$

where $K_x(f) = -d^2/dx^2 + U''(f(x))$ and $K_x(f)$ acts only on the first variable in the integrand of the first equation in (28).

We note that the system has no soliton solution with $E = E(f)$, since this would require in (28) that $f_0 = 0 = f_{01} = f_{21}$. However, there is the solution with $f_0 = f_{01} = f_{21} = 0$, $E = E(\phi)$, $f_{10} = \phi'$, $f_{21} = f_{22} = 0$; this corresponds to the previous 1-soliton, no-meson solution.

Let us now consider the solutions with $E \neq E(f)$.

We note that the system (28) must be treated with care when terms of given orders in $\hbar$ are to be compared. For as we can see from the last of Eqs. (28) the l.h.s. is $O(\hbar)$ whilst the r.h.s. is $O(1)$, coming from the term containing f_0. Substitution of this last equation into the first of those in (28), for $E - E(f) = \hbar\lambda$, $\lambda \neq 0$, gives

$$
\begin{aligned}
\left\{\lambda^2 - \tfrac{1}{2}\delta(0) \int [U''(f) - U''(\phi_0)]\, dx\right\} f_0 &= -\, 2\lambda\hbar \int dx\, K_x(f) f_{20}(xx) \\
&\quad + \tfrac{1}{2}i\hbar\delta(0) \int f_{10}(x)\, U'''(f)\, dx \\
&\quad - \tfrac{1}{2}\hbar^2\delta(0) \int f_{20}(xx)\, U^{(4)}(f)\, dx
\end{aligned}
\tag{29}
$$

$$
[K_x(f) - \lambda^2] f_{10}(x) = -i\hbar f_{20}(xx)\, U_x'''(f) \tag{30}
$$

$$
\begin{aligned}
[2K_x(f) + 2K_x(y) - \lambda^2] f_{20}(x, y) &= \frac{f_0}{\hbar\lambda}\, \delta(x - y)[U_x''(f) - U''(\phi_0)] \\
&\quad + \frac{i}{\lambda}\, \delta(x - y) f_{10}(x)\, U_x'''(f) \\
&\quad - \frac{\hbar}{\lambda}\, \delta(x - y) f_{20}(x, x)\, U_x^{(4)}(f)
\end{aligned}
\tag{31}
$$

$$f_{01}(x) = i\lambda f_{10}(x)$$

$$f_{21}(x, y) = i\lambda f_{20}(x, y)$$

$$f_{22}(x, y) = \frac{1}{2\hbar\lambda} f_0 \delta(x - y)[U''_x(f) - U''(\phi_0)] + \cdots \quad ((6), \text{p. } 13)$$

where $\delta(0)$ is to be interpreted by quantisation in a finite box.

We see that to lowest order in $\hbar$ these equations become

$$[\lambda^2 - \tfrac{1}{2}\delta(0)C] f_0 = 0 \tag{32}$$

$$[K_x(f) - \lambda^2] f_{10} = 0 \tag{33}$$

$$[2K_x(f) + 2K_y(f) - \lambda^2] f_{20}(x, y) = -\tfrac{1}{2}\delta(x - y) f_0 [U''(f) - U''(\phi_0)] \tag{34}$$

$$C = \int [U''(f) - U''(\phi_0)]\, dx.$$

We take $f_0 = 0$ in (32) to remove the infinite constant $\delta(0)$, so that the solutions of (33) and (34) are

(a) $$\lambda = \lambda_n\, , f_{10} = \phi_n\, , f_{20} = 0$$

or

(b) $$\lambda = [2(\lambda_n{}^2 + \lambda_m{}^2)]^{1/2}, f_{20} = \phi_n(x)\, \phi_m(y), f_{10} = f_{01} = 0.$$

The former class of solutions are the (one-meson + 1 soliton) states described in the previous section. The second type are clearly to be interpreted as (two-meson + 1 soliton) states, with threshold (2μ), as expected.

6. Higher Meson Approximations

It is evident from the previous section that the technique developed there will prove difficult to extend to the higher meson + 1 soliton states. Even more crucially there seems no appearance of quantum corrections to the soliton mass [1]. In order to remedy these defects we will use the Schrödinger field representation to set up the energy eigenvalue and eigenstate problem. We use the notation of the earlier section to solve the problem:

$$H(\pi, \phi) \mid E\rangle = E \mid E\rangle. \tag{35}$$

We take $\mid E\rangle = U^{-1}(\phi_c) \mid \psi\rangle$, where $\phi \mid \psi\rangle = \psi \mid \psi\rangle$, ψ being a classical field. Equation (35) becomes

$$H(\pi, \phi + \phi_c) \mid \phi\rangle = E \mid \phi\rangle.$$

Since in this representation $\pi = i\hbar\, \delta/\delta\phi$, we have to solve

$$\left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}\phi G^2\phi + V(\phi)\right] | \phi\rangle = [E - E(\phi_c)] \,|\, \phi\rangle \tag{36}$$

where

$$G^2(x, y) = \left[-\frac{d^2}{dx^2} + U''(\phi_c)\right] \delta(x - y) = \sum_m \phi_m(x)\, \omega_m{}^2 \phi_m(y) \tag{37}$$

$$V(\phi) = \sum_{n \geqslant 3} \frac{1}{n!}\, \phi^n U^{(n)}(\phi_c). \tag{38}$$

We note that the summation over m in (37) is only over $m > 0$, since $\omega_0 = 0$. We will have to take account of the lowest mode when G^{-1} is considered, but we will discuss that in due course. We take

$$| \phi\rangle = \exp[-\tfrac{1}{2}\phi G\phi] \,|\, x\rangle \tag{39}$$

$$E - E(\phi_c) = \lambda\hbar$$

to give

$$\left[-\frac{\hbar^2}{2}\frac{\delta^2}{\delta\phi^2} + \hbar G\phi \cdot \frac{\delta}{\delta\phi} + \frac{1}{2}\hbar \operatorname{tr} G + V(\phi)\right] | x\rangle = \delta\hbar \,|\, x\rangle. \tag{40}$$

Let us first solve (40) without the perturbation $V(\phi)$. If we expand

$$\begin{gathered} \phi = \sum_{n \neq 0} a_n \phi_n\,, \qquad \pi = \sum_{n \neq 0} \frac{\hbar}{i}\frac{\partial}{\partial a_n}\, \phi_n \\ G\phi = \sum_n \omega_n a_n \phi_n \\ | x\rangle = \chi(a_1\,, a_2\,, \cdots), \end{gathered} \tag{40a}$$

then (40) becomes

$$\left[\frac{\hbar^2}{2}\frac{\partial^2}{\partial a_n{}^2} - \hbar\omega_n a_n \frac{\partial}{\partial a_n} + \lambda_n \hbar\right] \chi = 0 \tag{41}$$

where $\sum \lambda_n = (\lambda - \frac{1}{2} \operatorname{tr} G)$. In terms of the dimensionless variables x_n, ϵ_n defined by $x_n = (\omega_n/\hbar)^{1/2} a_n$, $\lambda_n = \epsilon_n \omega_n$, (41) becomes the usual harmonic oscillator equation

$$\left[\frac{\partial^2}{\partial x_n{}^2} - 2x_n \frac{\partial}{\partial x_n} + \epsilon_n\right] \chi = 0$$

with solutions $\chi = \prod_i H_{m_i}(x_i)$, $\epsilon_n = m_i$, where the $H_n(x)$ is the nth Hermite polynomial and the m_i are integers. Thus the energy of the associated state is

$$\lambda\hbar = \frac{\hbar}{2} \operatorname{tr} G + \sum_i m_i \omega_{n_i} \hbar \tag{42}$$

the state itself being

$$\chi = \prod_i H_{m_i}(a_i(\omega_i/\hbar)^{1/2}) \tag{43}$$

where $a_i = \int \phi\phi_i \, dx$. We have the immediate interpretation of (42) and (43) as the energy of a state composed of one soliton and m_i mesons in the ith mode, each with energy $(\mu^2 + k_i^2)^{1/2}$ and momentum k_i. We note that (42) contains the zero-point energy $(\frac{1}{2}\hbar \sum_k \omega_k)$ as the first quantum correction to the energy of the single soliton state, for which $m_i = 0$ for all i. This agrees with other approaches [1]. We also note that the no-soliton contribution and meson mass renormalisation counter terms must be included, to give the well-known results [1].

We may extend the above discussion to all orders in the perturbing potential $V(\phi)$ of (38). The first-order perturbation expression for the energy will be

$$\langle\chi \mid V(\phi) \mid \chi\rangle = \int \prod_i H_{m_i}^2(x_i)\, dx_i\, e^{-\Sigma_i x_i^2} \sum_{m\geqslant 3} \int dx \left[\sum_n \left(\frac{\hbar}{\omega_n}\right)^{1/2} x_n\psi_n(x)\right]^m \frac{U^{(m)}}{m!}(\phi_c)$$

with similar expressions for the perturbation to the states (43). These expressions have the usual perturbation-theoretic interpretation, and can be given to arbitrary order.

We may also evaluate equal-time form factors, as

$$\langle E \mid \prod_{i=1}^n \phi(x_i) \mid E'\rangle = \langle\phi \mid \prod_{i=1}^n [\hat{\phi}(x_i) + \phi_c(x_i)] \mid \phi'\rangle$$

which can then be computed by means of (39) and (43). In the simplest case we have

$$\begin{aligned}\langle E \mid \phi(x) \mid E'\rangle &= \langle\phi \mid [\hat{\phi}(x) + \phi_c(x)] \mid \phi'\rangle \\ &= \phi_c(x)\langle\phi \mid \phi'\rangle + \langle\phi \mid \hat{\phi}(x) \mid \phi'\rangle\end{aligned}$$

Let us take $\mid \phi\rangle$ to be the one soliton state and $\mid \phi'\rangle$ to be the one meson plus one soliton state, so that

$$\begin{aligned}\phi(x_1 \cdots) &= e^{-\frac{1}{2}\Sigma x_i^2} \\ \phi'(x_1 \cdots) &= x_m e^{-\frac{1}{2}\Sigma x_i^2}.\end{aligned}$$

Then

$$\langle\phi \mid \phi'\rangle = 0,$$

and

$$\begin{aligned}\langle\phi \mid \hat{\phi}(x) \mid \phi'\rangle &= \int \prod dx_i\, H_1(x_m)\, e^{-\Sigma x_i^2}(\hbar/\omega_m)^{1/2}\, x_m\psi_m{}'x) \\ &= (\hbar/\omega_m)^{1/2}\, \psi_m(x).\end{aligned}$$

We thus obtain the result of Eq. (26) but with the extra correct normalisation factor $(\hbar/\omega_m)^{1/2}$; this agrees with the recent discussion of Steinmann [4] and earlier analysis [1]. We can furthermore obtain the 1-soliton plus mesons state expectation value of a product of unequal time fields by the same technique as used in Section 4, that of insertion of a complete set of intermediate states.

We can relate more closely to the discussion of the previous two sections by rewriting the functions $\chi(x_1 \cdots)$ in terms of the fields ϕ and $\dot{\phi}$. Thus we introduce the annihilation and creation operators $a^+(x)$, $a(x)$ respectively by

$$\begin{aligned} a^+(x) &= G^{-1/2}(\dot{\phi} + iG\phi)/(2\hbar)^{1/2} \\ a(x) &= G^{-1/2}(\dot{\phi} - iG\phi)/(2\hbar)^{1/2} \end{aligned} \tag{44}$$

so that the unperturbed Hamiltonian takes the usual harmonic oscillator form

$$\hbar \int dx\, dy\, a^+(x)\, G(xy)\, a(y).$$

We define the component creation and annihilation operators a_i^+, a_i by

$$a(x) = \sum_{i>0} a_i \phi_i(x), \qquad a_i = \int a(x)\, \phi_i(x)\, dx,$$

$$a^+(x) = \sum_{i>0} a_i^+ \phi_i^*(x), \qquad a_i^+ = \int a^+(x)\, \phi_i^*(x)\, dx,$$

so that the free Hamiltonian is

$$\sum_{i>0} \hbar\omega_i a_i^+ a_i .$$

We note that we define $G^{-1/2}$ in (44) by $\sum_{n>0} \psi_n^*(x)\, \omega_n^{-1/2} \psi_n(y)$, since the zero frequency mode is not required in the freefield Hamiltonian. The states $| E\rangle$ constructed earlier can be given in terms of the no-soliton vacuum state $| 0\rangle$ by

$$\begin{aligned} | E\rangle &= U(\phi_c) \prod_{i=1}^{n} (n_i!)^{-1/2} (a_i^+)^{n_i} \,| 0\rangle \\ &= U(\phi_c) \prod_{i=1}^{n} (n_i!)^{-1/2} \left[\int [G^{-1/2}(\dot{\phi} + iG\phi)/(2\hbar)^{1/2}]\, \phi_i^+\, dx\right]^{n_i} | 0\rangle. \end{aligned}$$

We thus have expressed the (1-soliton plus mesons) states in terms of the field operators at time $t = 0$ and so at any arbitrary time, if so desired, completing the results of the earlier sections. That $| 0\rangle$ is the no-soliton vacuum state is validated by the fact that the unperturbed field ϕ will have its ith component $\int \phi(x, t)\, \phi_i^*(x)\, dx$ developing in time with the factor $e^{i\omega_i t}$, so that a_i also annihilates the no-soliton vacuum state, for all i.

We have deliberately excluded the zero-frequency mode in the various summations we have used, especially in the expressions for ϕ and π in (40a). This is correct because the zero frequency mode would have lead to the $n = 0$ equation from (41):

$$\left(\frac{\partial^2}{\partial a_0^2} + \frac{2\lambda_0}{\hbar}\right) \chi = 0$$

which has no discrete spectrum in λ_0. We note that (40a) would appear to violate the CCR's, since from (40a)

$$[\pi(x), \phi(y)]_- = -i\hbar \sum_{n\neq 0} \phi_n(x)\, \phi_n(y) = -i\hbar[\delta^3(x-y) - \phi_0(x)\, \phi_0(y)]. \quad (45)$$

However, we may use the standard argument [1] to show that the last term in (45) is cancelled. Thus we compute the one-soliton matrix element of (45). The one-meson plus one soliton intermediate states give the contribution (45) and the single soliton intermediate state contribution cancels the last term in (45). Thus the expressions (40a) only describe the one-soliton plus one meson sectors, and do not completely describe the one soliton no-meson state; this requires the additional coherent state field operator $U(\phi_c, t)$ and thus the corresponding wavefunction $\phi_c(x)$.

We finally show that the soliton is absolutely stable against decay into any finite number of mesons. In the Schrödinger representation we have that the 1 soliton and no-soliton states are respectively

$$|\,\phi_c\rangle = \exp\left[-\frac{1}{2\hbar}(\phi + \hat{\phi}_c)\, G(\phi + \hat{\phi}_c)\right]$$

$$|\,\phi_0\rangle = \exp\left[-\frac{1}{2\hbar}\, \phi G_0 \phi\right]$$

where

$$G_0 = \left[-\frac{d^2}{dx^2} + U''(\phi_0)\right] \delta(x-y)$$

with plane-wave eigenstates $\bar{\psi}_i$. Then the overlap $\langle \phi_c \mid \phi_0\rangle$ has as integrand

$$-\tfrac{1}{2} \sum{}' (y_i + a_i)(y_j + a_j)\, G_{ij} - \tfrac{1}{2} \sum y_i^2$$

where

$$\phi = \sum_i (\hbar/\omega_i)^{1/2}\, y_i \bar{\psi}_i$$

$$\hat{\phi}_c = \sum_i (\hbar/\omega_i)^{1/2}\, a_i \bar{\psi}_i$$

$$G_{ij} = \frac{\hbar}{(\omega_i \omega_j)^{1/2}} \int \bar{\psi}_i^*(x)\, G(x, y)\, \bar{\psi}_j(y)\, dx\, dy.$$

A change of variables to $z = G^{1/2} y$ leads to an integrand including the factor $-\sum_i a_i^2$ in the exponent. But this has the value

$$-\hbar^{-1} \int |\, \tilde{\hat{\phi}}_c(k)|^2\, \omega_k\, dk$$

which has already been shown to be $-\infty$ corresponding to the existence of a topological conservation law $\hat{\phi}(\pm\infty) \neq 0$. Thus the stability result follows.

7. Time Dependence

We have so far restricted our discussion to purely time-independent solutions of the classical field equations. Let us now consider how time-dependence may be accommodated in the Hamiltonian formalism. We will do that for the case of a moving soliton, with wavefunction $f(x, t) = \phi_c(\gamma(x - vt))$, where $\gamma = (t - v^2)^{-1/2}$, so that f satisfies the time-dependent wave equation

$$\ddot{f} - f'' + U(f) = 0$$

and has energy $\int dx[\frac{1}{2}\dot{f}^2 + \frac{1}{2}f'^2 + U(f)] = \gamma \ E(\phi_c)$, as expected. Then the first approximation to the moving soliton state will be obtained by application of the coherent state operator $U(f, t, u) = \exp(i/\hbar)M(f, t, u)$ similar to (5), but with $M(f, t, u)$ now defined by

$$M(f, t, u) = \int dx \ \hat{\phi}_c(\gamma(x - ut)) \ \partial_0\phi(x, t).$$

Furthermore,

$$U(f, t, u)\phi(x, t) \ U^{-1}(f, t, u) = \hat{\phi}(x, t) + f(\gamma(x - ut))$$
$$U(f, t, u)\dot{\phi}(x, t) \ U^{-1}(f, t, u = \dot{\phi}(x, t) + \dot{f}(\gamma(x - ut))$$

so then

$$U(f, t, u)H(\pi, \phi) \ U^{-1}(f, t, u) = \gamma E(\phi_c) + (\text{terms containing } \hat{\phi}, \dot{\phi}).$$

Thus the state $| \ \phi_c , t, u\rangle$ is the lowest approximation to a single soliton with velocity u and energy $\gamma E(\phi_c)$. It may be translated to be at the point y at time 0 by means of the translation operator $\exp((i/\hbar) \ Py)$, to give the state $| \ \phi_c , t, y, u\rangle$,

$$| \ \phi_c , t, y, u\rangle = \exp((i/\hbar) \ Py) \ | \ \phi_c , t, u\rangle$$

Finally Fourier transformation on y will produce the state $| \ \phi_c , t, p, u\rangle$:

$$| \ \phi_c , t, p, u\rangle = \int dy \ e^{ipy/\hbar} \ | \ \phi_c , t, y, u\rangle.$$

We may use this construction to consider the question of asymptotic states and fields for the soliton. To do that we will take the soliton state of given momentum, $| \ \phi_c , p\rangle$, defined by

$$| \ \phi_c , t, p, v\rangle = | \ \phi_c , p\rangle \ e^{-ip_0t/\hbar} \tag{46}$$

with $v = p\gamma/E(\phi_c)$, this state has momentum p and energy $p_0 = \gamma E(\phi_c)$ so that we have the soliton mass shell condition

$$p_0^2 - p^2 = E^2(\phi_c).$$

We may obtain the state (46) by applying to the vacuum the operator

$$\int dy\, e^{ipy/\hbar} e^{iPy/\hbar} U^{-1}(\phi_c, t, p\gamma/E(\phi_c)) = B(p, t) \tag{47}$$

and form the inverse transform

$$B(x, t) = \int e^{-ipx} B(p, t)\, dp.$$

From Eqs. (46) and (47) we see that

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + E^2(\phi_c)\right] B(x, t) \mid 0\rangle = 0.$$

Thus we see that $B(x, t)$ is an appropriate operator to use to define asymptotic fields [3]:

$$B_{\text{in}}(x, t) = B(x, t) + \int \Delta_{\text{ret}}(E(\phi_c), x - x', t - t')\, j(x', t')\, dx'\, dt'$$

$$B_{\text{out}}(x, t) = B(x, t) + \int \Delta_{\text{adv}}(E(\phi_c), x - x', t' - t')\, j(x', t')\, dx'\, dt'$$

where

$$j(x, t) = [\Box + E^2(\phi_c)] B(x, t).$$

Then the asymptotic fields $B_{\text{in}}(x, t)$, $B_{\text{out}}(x, t)$ are obtained by the usual asymptotic limits from $B(x, t)$ and can be used to construct S-matrix elements involving one external soliton. In particular the usual reduction formulas will be valid, giving S-matrix elements in terms of

$$\prod_{i=1}^{n} K_{x_i} \prod_{j=n+1}^{n+2} [K_{x_j} - E^2(\phi_c)] \langle 0 \mid T(\phi(x, t_1) \cdots \phi(x_n t_n)\, B(x_{n+1} t_{n+1})\, B(x_{n+2} t_{n+2})) \mid 0\rangle.$$

We cannot expect reduction formulas to apply to give multisoliton states in this manner, since there may not be such states in the spectrum of the Hamiltonian at the classical level.

8. Discussion

We have tried to present evidence for the thesis that a soliton should be regarded as constructed from a suitable coherent state of the original meson field in the no-soliton sector. The evidence for this is strong in the case of static solutions to the classical field equations but is not so supportive for time dependent solutions involving multisoliton states. The collective excitation method [1] seems more appropriate in this case. However there appears no reason in principle why the physical features of the single soliton state considered here should not be valid for the more general case. We will discuss the physical implications of our results from that point of view.

What is claimed to have been shown is that a soliton is an infinitely constituent 'bound state'. The original constituents are bound by means of the coherence provided by the solution of the original classical equations of motion. Thus the soliton states are not bound states in the usual sense of being constructed out of a finite number of fundamental constituents bound together by a suitable potential. Yet they are constructed solely from the mesons of the no-soliton sector.

It is this feature which is most intriguing. Its detailed expression is given in terms of the classical field solutions describing the various form factors of the soliton states. In order to test such aspects it will be necessary to have a realistic model of the elementary particles whose classical solutions are to be obtained. Thus traditional bound state conditions, such as $Z = 0$ [5], appear difficult to use immediately for detecting which particles might be solitons.

One general result of a bound-state nature that does follow from our result is that the field theory from which solitons arose will not have any amelioration of its ultraviolet or infrared divergences. This is clear in the no-soliton sector, while solitons may only be satisfactorily treated if the higher quantum corrections to their various physical quantities are all finite. This is similar to the situation for traditional bound states. The extra asymptotic states to which the solitons and their meson clouds correspond can be treated as independent 'elementary' particles with associated fields and interactions. Such a description could be built out of the original Hamiltonian by means of suitable coherent state operators. But the 'elementary' nature of the solitons would be as much of an illusion as that of traditional bound states, even though they may be treated on the same level as their elementary constituents for certain features of their dynamics, such as for deriving reduction formulas and dispersion relations [3].

The most important aspect of the soliton is its complete confinement of its constituents. This attractive feature, as well as the large soliton mass, leads to the conjecture of constructing baryons from more weakly interacting particles. The most natural of these is the lepton family. A recent attempt has been made to construct such a model [6], though not using the soliton concept. There are various crucial difficulties about such a program, such as achieving a low enough magnetic moment from leptonic constituents, and also of localising such light constituents. It is clearly necessary to investigate these features, as well as the symmetry aspects, further. We hope to turn to the identification of quarks as leptons elsewhere.

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